# Ideal analytic sets

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# Analytic complete sets

**Definition**

Let $A \subseteq X$, $B \subseteq Y$. We say, that $B$ is *Borel reducible* to $A$ if there exists a Borel map $f : Y \to X$ such that $f^{-1}[A] = B$.

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**Remark**

If analytic set $B$ is Borel reducible to $A$ and $B$ is $\Sigma^1_1$-complete, then $A$ is also $\Sigma^1_1$ complete.

# Ideals on $\omega$

For defining ideals we will use approach from [1].

## Definition

Let $\mathcal{F} \subseteq [\omega]^\omega$. For a function $\rho : \mathcal{F} \to [\omega]^\omega$ we define a family

$$\mathcal{I}_\rho = \{A \subseteq \omega : (\forall F \in \mathcal{F})(\rho(F) \nsubseteq A)\}.$$

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If $\mathcal{I}$ is an ideal on $\omega$, then it admits a function

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**Fact (Filipów, Kowitz, Kwela)**

*If $\rho$ is continuous and $\mathcal{F}$ is closed, then the ideal $\mathcal{I}_\rho$ is coanalytic.*

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3. Note that if $\tau \in [T]$, then
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4. Prove that if $\rho(F) \subseteq f(T)$ for some $F \in \mathcal{F}$, then $[T] \neq \emptyset$.

# Bad news

**Proposition (folklore)**

The summable ideal

$$\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty \right\}$$

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**Proposition (folklore)**

A set $A \subseteq \omega$ is called an AP-set if it contains an arithmetic progressions of arbitrary finite length. The van der Waerden ideal

$$\mathcal{W} = \{A \subseteq \omega : A \text{ is not an AP-set}\}$$

is $F_\sigma$.

# Good news

## Theorem (M., Żeberski)

*Following ideals are $\mathbf{\Pi}^1_1$-complete:*

- *Ramsey ideal $\mathcal{R} = \mathcal{I}_r$, where $r(F) = [F]^2$,*
- *Hindman ideal $\mathcal{H} = \mathcal{I}_{FS}$, where*

$$FS(F) = \left\{ \sum_{n \in B} n : B \in [F]^{<\omega} \setminus \emptyset \right\},$$

- *ideal $\mathcal{H}_2 = \mathcal{I}_{FS_2}$, where*

$$FS_2(F) = \left\{ \sum_{n \in B} n : B \in [F]^2 \right\},$$

- *ideal $\mathcal{D} = \mathcal{I}_\Delta$, where*

$$\Delta(F) = \{a - b : a, b \in F, a > b\}.$$

# Connection with trees

**Theorem (M., Żeberski)**

*The family $\mathcal{I}_M$, given by*

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A tree $T \in \mathrm{Tree}_\omega$ is Mathias if there is $\sigma \in T$ and $F \in [\omega]^\omega$ such that

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## Corollary

*A family of all trees containing a Mathias tree is $\mathbf{\Sigma}^1_1$-complete.*

# References


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# Thank You for attention

And we invite you to 2nd Wrocław Logic Conference